A note on Price's equation for evolutionary economics: Derivation, interpretations, and simple applications

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1. Introduction

The breakthrough in formal modelling of individual populations and their evolution was largely made by Fisher (1999[1930]). A core result was the Fisher theorem that '[t]he rate of increase of fitness of any species is equal to the genetic variance in fitness' (Fisher, 1999[1930], 46). The discussion of this theorem included an emphasis on further studies of other elements of the evolutionary process than the selection from given variance. Since the narrow theorem has created some unwarrented interpretations of Fisher's work, Metcalfe (1994) suggests that we should emphasise the Fisher principle (which includes the broader issues) rather than the narrow Fisher theorem. He and other evolutionary economists have tried to develop this principle formally (see e.g. Metcalfe, 1998, 2001). Later, Metcalfe (2002, 90) has found out that '[f]or some years now evolutionary economists have been using the Price equation without realising it.' A few have made the same discovery as Metcalfe. Thus, several game theorists have begun to apply Price's equation (cf. Gintis, 2000, 267?268), and Knudsen (2002) have used both the equation and Price's general account for selection to rethink the role of habits and routines in theories of economic evolution.

To overcome some of the ambiguities of Fisher's formulation of his theorem, Price (1970; 1972) made a decomposition of the evolutionary change that included not only the effect of selection but also the effect of causes that increases variation. Price's equation or equation is not easy to understand, so even though it resolves many of Fisher's problems, it is often used in a delimited version that is of less importance. Frank (1995; 1997; 1998) has been a major contributor to the development and diffusion of the full version of Price's decomposition of evolutionary change. His contributions demonstrate that a large number of evolutionary problems can be clarified by means of Price's equation. They also make clear that many researchers have been moving in the same direction as Price without noticing the full generality of their results and their relationship to Price.

2. Price's equation: Context and derivation

2.1 Definitions for the analysis

To grasp Price's equation we have to apply population thinking in a rather advanced way (alternative accounts are e.g. found in Gintis, 2000, and Knudsen, 2002). To begin with, we think of a population that can be subdivided into components (subpopulations). These components may be smaller populations or basic units that cannot be adequately subdivided (individuals or, perhaps, firms). The same equation applies irrespectively of how we split the overall population into components.

The population is subdivided into n components (of at least one basic unit). Each subpopulation has the size of n_i (measured in basic units).

Population share of component i is

$$s_i = n_i/n,$$

while mean population share is

$$\bar{s} = \sum s_i = 1.$$

The quantative characteristic of firm i is denoted by A_i. The expected value of the characteristic of a sample drawn from the population (or its mean characteristic) is by standard definition

$$\mathrm{E}(A_i) = \bar{A} = \sum s_i A_i.$$

Similarly, variance of characteristics is by standard definition

$$\mathrm{Var}(A_i) = \sum s_i (A_i - \overline{A})^2.$$

2.2 Change

Price's equation is about comparing the variables between two points of time, and we shall denote the values of the variables at the first point of time with their ordinary names and at the second point of time with prime marks.

The purpose of taking two points of time is to find out how well components reproduce, i.e. relation between the size of the component at the two points of time. This reproduction coefficient is called absolute fitness. The fitness of a component may be influenced by a large number of its characteristics. However, we concentrate on one characteristic of the members of the component, and the problem first is to find how fitness relates to that characteristic.

The new population size is n'_i, and its change is

$$\Delta n_i = n'_i - n_i,$$

and the change in population share of a subpopulation is

$$\Delta s_i = s'_i - s_i.$$

Reproduction coefficient (absolute fitness) is

$$r_i = n'_i / n_i,$$

while mean of reproduction coefficients is

$$\overline{r} = \sum s_i r_i.$$

On this background we see that the new population share

$$s'_i = n'_i / n' = (s_i r_i)/(\overline{s}\,\overline{r}) = s_i r_i / \overline{r}.$$

Change in characteristics of a subpopulation is

$$\Delta A_i = A'_i - A_i.$$

Covariance between reproduction coefficients and characteristics is by standard definition

$$\mathrm{Cov}(r_i, A_i) = \sum s_i (r_i - \overline{r})(A_i - \overline{A}) = \beta_{r_i A_i} \mathrm{Var}(A_i).$$

2.3 Price's equation in two versions

Price's equation is an identity and thus universally applicable. The two versions of Price's equation are normally expressed as

$$\begin{aligned} \overline{r}\Delta\overline{A} &= \mathrm{Cov}(r_i, A_i) + \mathrm{E}(r_i \Delta A_i), \\ &= \beta_{r_i A_i} \mathrm{Var}(A_i) + \mathrm{E}(r_i \Delta A_i). \end{aligned} \tag{1}$$

The term on the left hand side of Price's equation might look mysterious. By dividing both sides of Price's equation by $\overline{r}$, we obtain more a format that is more easily interpreted: Price equation is simply about $\Delta\overline{A}$. The standard format has typographical advantages. The main reason why the format is chosen will, however, become clear in section 2.6.

2.4 Derivation of Price's equation

The Price equation may derived in the following way:

$$\begin{aligned}
\Delta\overline{A} &= \overline{A}' - \overline{A} \\
&= \sum s_i' A_i' - \sum s_i A_i \\
&= \sum s_i \frac{r_i}{\overline{r}}(A_i + \Delta A_i) - \sum s_i A_i \\
&= \sum s_i \frac{r_i}{\overline{r}} A_i + \sum s_i \frac{r_i}{\overline{r}} \Delta A_i - \sum s_i A_i \\
&= \sum s_i (\frac{r_i}{\overline{r}} - 1) A_i + \sum s_i \frac{r_i}{\overline{r}} \Delta A_i \\
&= \sum s_i (\frac{r_i - \overline{r}}{\overline{r}}) A_i + \sum s_i \frac{r_i}{\overline{r}} \Delta A_i \\
&= \frac{1}{\overline{r}} \sum s_i (r_i - \overline{r}) A_i + \frac{1}{\overline{r}} \sum s_i r_i \Delta A_i
\end{aligned}$$

Because the expression $1/\overline{r} \sum s_i (r_i - \overline{r})\overline{A}$ is zero (see below), we may insert it:

$$\begin{aligned}
&= \frac{1}{\overline{r}} \sum s_i (r_i - \overline{r}) A_i - \frac{1}{\overline{r}} \sum s_i (r_i - \overline{r}) \overline{A} + \frac{1}{\overline{r}} \sum s_i r_i \Delta A_i \\
&= \frac{1}{\overline{r}} \sum s_i (r_i - \overline{r})(A_i - \overline{A}) + \frac{1}{\overline{r}} \sum s_i r_i \Delta A_i
\end{aligned}$$

By applying the standard definitions for the covariance between r_i and A_j and the expected value of the product $r_i \Delta A_i$, we obtain the first version of the right hand side of Price's equation:

$$= \frac{1}{\overline{r}} \mathrm{Cov}(r_i, A_i) + \frac{1}{\overline{r}} \mathrm{E}(r_i \Delta A_i)$$

The covariance may also be expressed by the product of the regression coefficient of r_i on A_i and the variance of A_i. Thus we have a second version of the right hand side of Price's equation:

$$= \frac{1}{\overline{r}} \beta_{r_i A_i} \mathrm{Var}(A_i) + \frac{1}{\overline{r}} \mathrm{E}(r_i \Delta A_i).$$

We have now derived two alternative forms of Price's equation. During this derivation we exploited the following result:

$$\begin{aligned}
\frac{1}{\overline{r}} \sum s_i (r_i - \overline{r}) \overline{A} &= \frac{1}{\overline{r}\overline{\overline{A}}} \sum s_i (r_i - \overline{r}) \\
&= \frac{1}{\overline{r}\overline{\overline{A}}} \sum s_i r_i - \frac{1}{\overline{r}\overline{\overline{A}}} \sum s_i \overline{r} \\
&= \frac{\overline{r}}{\overline{r}\overline{\overline{A}}} - \frac{\overline{r}}{\overline{r}\overline{\overline{A}}} \sum s_i \\
&= \frac{1}{\overline{\overline{A}}} - \frac{1}{\overline{\overline{A}}} \\
&= 0.
\end{aligned}$$

2.5 Interpreting Price's equation

Let us quickly consider the main elements of equations 1:
$\mathrm{Cov}(r_i, A_i)$ may be interpreted as the selection effect.
$\beta_{r_i A_i}$ is the efficiency of this selection.
$\mathrm{Var}(A_i)$ is the variance that selection has to work on.
$\mathrm{E}(r_i \Delta A_i)$ is potentially very complex.

If each subpopulation consists of a basic unit, then $\mathrm{E}(r_i \Delta A_i)$ has a (positive or negative) 'innovation effect'. This effect is a mix of the environment's effect on the characteristics and the change within the basic units.

If some subpopulations consist of several subpopulations, then $\mathrm{E}(r_i \Delta A_i)$ may be decomposed into a subpopulation-internal selection effect and a subsubpopulation 'innovation effect'. This will be considered in section 2.6.

2.6 Multi-level decomposition by Price's equation

Equation 1 has the interesting property that the left hand side ($\overline{r}\Delta\overline{A}$) is equal to the product for which the mean is found ($r_i \Delta A_i$)—except for the fact that the latter is taken for individual firms. But the equation does not depend on this specific interpretation. In the case where the right hand side is dealing with groups, the recursive structure of Price's equation can be exploited (Price, 1972; Hamilton, 1975; Grafen, 1985). So we shall assume some grouping of firms, e.g. according to locality.

To understand the possibilities implied by this grouping, we start by decomposing the population of firms into groups indexed by g, and then we identify the firms by group with the index gi. Let us rewrite equation 1 in this notation:

$$\overline{r}\Delta\overline{A} = \mathrm{Cov}(r_g, A_g) + \mathrm{E}(r_g \Delta A_g). \tag{2}$$

Now each group represents a local population of firms for which

$$r_g = \overline{r_g} = \sum s_{gi} r_{gi}$$

and

$$A_g = \overline{A_g} = \sum s_{gi} A_{gi}.$$

Thus we can use Price's equation (1) to decompose productivity change within each group:

$$r_g \Delta A_g = \mathrm{Cov}(r_{gi}, A_{gi}) + \mathrm{E}(r_{gi} \Delta A_{gi}). \tag{3}$$

This result can be substituted for $r_g \Delta A_g$ in equation 2:

$$\overline{r}\Delta\overline{A} = \mathrm{Cov}(r_g, A_g) + \mathrm{E}\left[\mathrm{Cov}(r_{gi}, A_{gi}) + \mathrm{E}(r_{gi} \Delta A_{gi})\right]. \tag{4}$$

Here the individual version of Price's equation (3) is nested in the expectation part of the group version of Price's equation (2).

This recursive decomposition demonstrates that what might be thought of as an intra-group 'innovation' effect is really the outcome of a combination of an intra-group selection effect and an intra-firm 'innovation' effect. Thus it is possible to analyse the functioning of two different processes of selection: selection between the groups of an economy and selection between firms at the group level. Thus we may obtain new insights to the extent that these selection forms works differently, i.e. if

$$\beta_{r_g A_g} \neq \beta_{r_{gi} A_{gi}}$$

for some groups.

This issue will be touched upon in section 3.5. Presently we should just note that if new groups emerge between t and t', then they are ascribed to their mother group.

3. Example: The *AL* model and Price's equation

3.1 Introduction

In section 1 it was pointed out that Nelson and Winter (1982, 242–245) made a decomposition of change of productivity. This decomposition can be shown to be a an independent discovery of Price's equation for the special case of perfect selection. Nelson and Winter's rediscovery of Price's equation was made in the context of a growth model, where there are both capital productivities and labour productivities. To demonstrate quickly that their result is a special version of Price's equation it is, however, convenient to apply a model in which there is only labour. This model is the *AL* model, which is developed by Andersen (forthcoming) (in a pure labour form).

3.2 Production conditions

We shall consider a pure labour economy in which a population of firms produce output by means of their knowledge A_i and labour L_i. The knowledge of the firm is simply represented as a productivity, which in a disembodied way is shared by both old and new labour. The firm's production takes place according to the Leontief production function

$$Q_i = A_i L_i,$$

while aggregate output is

$$Q = \sum Q_i.$$

We also need to know aggregate labour

$$L = \sum L_i$$

and labour shares

$$s_i = \frac{L_i}{L}.$$

With respect to the productivities, we need mean productivity

$$\overline{A} = \sum s_i A_i$$

and variance of productivities

$$\mathrm{Var}(A_i) = \sum s_i (A_i - \overline{A})^2.$$

Since

$$\begin{aligned} Q &= \sum Q_i \\ &= \sum A_i L_i \\ &= \frac{L}{L} \sum A_i L_i \\ &= L \sum A_i \frac{L_i}{L} \\ &= L \sum A_i s_i \\ &= L\overline{A}, \end{aligned}$$

we are sure about the obvious fact that

$$Q = \overline{A}L.$$

3.3 Change in the *AL* model

Labour is owned by households. For each period it is hired by firms for one monetary unit per unit of labour. All household income is spent on the output of the firms. Thus the market clearing price

$$\begin{aligned} P &= \frac{D}{Q} \\ &= \frac{L}{\overline{A}L} \\ &= \frac{1}{\overline{A}}. \end{aligned}$$

At this price the revenue of a firm is PQ_i. Since the wage rate is unity, the profit rate

$$\begin{aligned}\pi_i &= \frac{PQ_i - L_i}{L_i}\\ &= \frac{PA_iL_i - L_i}{L_i}\\ &= PA_i - 1\\ &= \frac{A_i}{\overline{A}} - 1.\end{aligned}$$

Firms spend all revenues on labour. Thus the change of labour

$$\begin{aligned}\Delta L_i &= \pi_i L_i\\ &= (\frac{A_i}{\overline{A}} - 1)L_i\\ &= (\frac{A_i}{\overline{A}} - \frac{\overline{A}}{\overline{A}})L_i\\ &= (\frac{A_i - \overline{A}}{\overline{A}})L_i.\end{aligned}$$

This is the famous replicator dynamic equation. It implies that firms have zero change if their productivity is equal to the mean productivity. Firms with overnormal productivity increase labour, while subnormal firms decrease labour.

This simple replicator dynamic is based on the fact that the mean profit rate

$$\begin{aligned}\overline{\pi} &= \sum s_i \pi_i\\ &= \sum \frac{s_i(A_i - \overline{A})}{\overline{A}}\\ &= \frac{1}{\overline{A}} \sum s_i(A_i - \overline{A})\\ &= \frac{1}{\overline{A}} \sum s_i A_i - \frac{1}{\overline{A}} \sum s_i \overline{A}\\ &= \frac{\overline{A}}{\overline{A}} - \frac{A}{\overline{A}} \sum s_i\\ &= 1 - 1 \sum s_i\\ &= 1 - 1\\ &= 0.\end{aligned}$$

This result implies that aggregate employment L does not change. This, of course, is not true in more complex models. We shall, however, ignore this problem (cf. Metcalfe, 1994).

Let us now consider change in labour share

$$\begin{aligned}\Delta s_i &= s_i' - s_i\\ &= \frac{L_i'}{L} - \frac{L_i}{L}\\ &= \frac{L_i' - L_i}{L}\\ &= \frac{L_i + \pi_i L_i - L_i}{L}\\ &= \frac{\pi_i L_i}{L}\\ &= \pi_i s_i\end{aligned}$$

or

$$s'_i = (1 + \pi_i)s_i.$$

A simple replicator dynamic assumes that productivities do not change over time, but this cannot be assumed in general. Therefore, we shall introduce

$$\Delta A_i = A'_i - A_i.$$

This change in productivity may be caused by many mechanisms, which we presently shall not explore. However, we may characterise the aggregate result of this change by the mean change in productivity

$$\Delta \overline{A} = \sum s_i \Delta A_i.$$

3.4 Applying Price's equation to the *AL* model

To bring the *AL* model into a format which is equivalent to Price's equation, we need to add the reproduction coefficient of labour

$$r_i = \frac{L'_i}{L_i},$$

but we have not met this variable in the previous account for the *AL* model. However, we see that

$$\begin{aligned} r_i &= \frac{L'_i}{L_i} \\ &= \frac{L_i + \Delta L_i}{L_i} \\ &= \frac{L_i + \pi_i L_i}{L_i} \\ &= 1 + \pi_i. \end{aligned}$$

Given this result, we see that the mean reproduction coefficient

$$\begin{aligned} \overline{r} &= \sum s_i r_i \\ &= \sum s_i (1 + \pi_i) \\ &= \sum s_i + \sum s_i \pi_i \\ &= 1 + \overline{\pi} \\ &= 1. \end{aligned}$$

Let us now consider the covariance between reproduction coefficients and productivities

$$\begin{aligned} \mathrm{Cov}(r_i, A_i) &= \sum s_i (r_i - \overline{r})(A_i - \overline{A}) \\ &= \sum s_i (1 + \pi_i - 1)(A_i - \overline{A}) \\ &= \sum s_i \pi_i (A_i - \overline{A}) \\ &= \sum s_i \left(\frac{A_i - \overline{A}}{\overline{A}} \right) (A_i - \overline{A}) \\ &= \frac{1}{\overline{A}} \sum s_i (A_i - \overline{A})^2 \\ &= \frac{1}{\overline{A}} \mathrm{Var}(A_i). \end{aligned}$$

Given these results, we apply Price's equation (1) to decompose productivity change in the AL model. Because $\overline{r} = 1$, we simply decompose change in mean productivity

$$\begin{aligned} \mathrm{E}(r_i \Delta A_i) &= \mathrm{E}((1+\pi_i)\Delta A_i) \\ &= \mathrm{E}((1+\frac{A_i}{\overline{A}}-1)\Delta A_i) \\ &= \mathrm{E}((1+\frac{A_i}{\overline{A}}-1)\Delta A_i) \\ &= \mathrm{E}(\frac{A_i \Delta A_i}{\overline{A}}) \\ &= \frac{1}{\overline{A}}\mathrm{E}(A_i \Delta A_i). \end{aligned}$$

By inserting these results in Price's equation (1), we see that

$$\begin{aligned} \overline{r}\Delta\overline{A} &= \beta_{r_i A_i}\mathrm{Var}(A_i) + \mathrm{E}(r_i \Delta A_i) \\ \overline{r}\Delta\overline{A} &= \frac{1}{\overline{A}}\mathrm{Var}(A_i) + \frac{1}{\overline{A}}\mathrm{E}(A_i \Delta A_i). \end{aligned}$$

In our simple context the selection effect is determined by the regression coefficient $1/\overline{A}$ times variance of the productivities. The innovation effect is determined by the expected value of the product of productivities and change in productivities. Let us consider the latter effect:

$$\begin{aligned} \frac{1}{\overline{A}}\mathrm{E}(A_i \Delta A_i) &= \frac{1}{\overline{A}}\sum s_i A_i \Delta A_i \\ &= \frac{1}{\overline{A}}\sum \frac{L_i}{L} A_i \Delta A_i \\ &= \frac{1}{\overline{A}L}\sum L_i A_i \Delta A_i \\ &= \frac{1}{Q}\sum Q_i \Delta A_i. \end{aligned}$$

Thus we see that there are two determinants of the size of the innovation effect. The first concerns the relationship between innovation and the output of the firms. If innovations mainly takes place in small firms, then it does not matter much. If it is concentrated in large firms, then it is of larger importance. The second determinant is the size of the output weighted mean relative to the total output.

To sum up, we have under the assumptions of the simple AL model fund a special version of Price's equation, where

$$\overline{A}\Delta\overline{A} = \mathrm{Var}(A_i) + \mathrm{E}(A_i \Delta A_i). \tag{5}$$

3.5 Recursive use of Price's equation in the AL model

Equation 5 has the same property as the general Price equation (see section 2.6): that the left hand side ($\overline{A}\Delta\overline{A}$) is structurally equal to the product for which the mean is found ($A_i \Delta A_i$). By introducing a grouping of firms (e.g. according to locality), the recursive structure of Price's equation can be exploited even here.

Let us quickly repeat the results from section 2.6. To make the equation 5 in this notation:

$$\overline{A}\Delta\overline{A} = \mathrm{Var}(A_g) + \mathrm{E}(A_g \Delta A_g).$$

Then we decompose productivity change within each group:

$$A_g \Delta A_g = \mathrm{Var}(A_{gi}) + \mathrm{E}(A_{gi}\Delta A_{gi}).$$

Finally we insert the intra-group result into the inter-group result:

$$\overline{A}\Delta\overline{A} = \mathrm{Var}(A_g) + \mathrm{E}\left[\mathrm{Var}(A_{gi}) + \mathrm{E}(A_{gi}\Delta A_{gi})\right]. \tag{6}$$

This result is formally correct, but rather uninteresting. The problem is that it is exactly the same selection intensity that is found at the intra-group and the inter-group level. Thus the two selection environments are identical and we obtain no new insight by the introduction of groups. To make a grouping of firms relevant, we have to go beyond the simple structure of the *AL* model.

Acknowledgements The research underlying this paper was started by a suggestion by Stan Metcalfe to the author to have a closer look at Frank (1998). Inspiration was also provided by Thorbjørn Knudsen's comments on Andersen (2001). Important comments on the present paper has been given by Toke Reichstein.

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